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On symmetry reductions of topological/cohomological Yang–Mills theories

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Abstract. General aspects of symmetry reductions of topological/cohomological Yang–Mills theories are discussed. It is shown that cohomological quantum field theories associated with the Nahm equations can be derived from the Donaldson–Witten theory by a symmetry reduction with respect to a three-dimensional Abelian group.

1. Introduction

Supersymmetric Yang–Mills theories have appeared in the study of Dirichlet p-branes, or D-branes, which have been useful in the description of non-perturbative behaviour of string theory [1–3]. It is known that the low-energy description of n D-branes in flat space can be obtained via the dimensional reduction of (Euclidean) $D = 10$, $N = 1$ super-Yang–Mills (SYM) systems to $p+1$ dimensions [1–5]. Such supersymmetric theories have also been related to special types of theories, the topological or cohomological quantum field theories [6–14], which arise in a twisted form in the study of the moduli spaces of solutions of certain (nonlinear) equations such as the (anti-)self-dual Yang–Mills (SDYM) equations.

The moduli space of (anti-)SDYM equations in four dimensions is known to be described in terms of topological invariants (polynomials) through a quantum field theoretic system related to twisted $D = 4$, $N = 2$ SYM theories [15–20]. It has also been shown that the Seiberg–Witten equations (or Abelian monopole equations [20–23]) lead to a description of the Donaldson invariants. A generalization to non-Abelian monopole equations via the addition of coupled $N = 2$ hypermultiplet matter [24, 25] has been achieved and includes the above formulations (see also [26]). Higher-dimensional formulations of SDYM equations have been written [27–30]. For some of them, it has been found that their moduli space is probed by cohomological quantum field theories (cf, for example, [8–10]). Topological/cohomological field theories have also been useful in recent different works (cf, for example, [11, 12]).

Dimensional reductions of the Donaldson–Witten theory have been carried out to generate topological field theories in three dimensions [31, 32] including as a limit the $SO(3)$ Georgi–Glashow model, as well as in two dimensions [33] with the Teichmüller space of compact Riemann surfaces as moduli space. The dual theory associated with the Seiberg–Witten equations has also been dimensionally reduced to Abelian theories in dimensions two and three [34, 35] which are believed to be dual to the corresponding above-mentioned dimensional reductions of the Donaldson–Witten theory.

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Higher-dimensional cohomological field theories can be similarly reduced [8–10]. For instance, an eight-dimensional cohomological field theory [8] is related to a twisted version of $D = 4$, $N = 4$ SYM theory which is associated with non-Abelian Seiberg–Witten equations. Intermediate-dimensional reductions have also been considered leading to cohomological theories in dimensions 4–7. A different type of reduction has been attempted to retrieve two-dimensional theories by allowing a splitting of a four-dimensional space M into the (Cartesian) product of two Riemannian surfaces: $M = \Sigma \times C$, and by letting the metric of one of the surfaces vanish through scaling (i.e. as in compactification) [36,37]. For instance, under certain restrictions, one is led to supersymmetric σ-models on the moduli space of flat connections on C (for $N = 2$) or on the Hitchin space (for $N = 4$). Compactifications of $N = 4$ twisted SYM theories are also discussed in [38].

In terms of the field content of a quantum theory, the procedure of dimensional reduction consists mainly, for each subtracted dimension, in eliminating one natural coordinate and then ignoring any dependence of the field variables with respect to this coordinate [8]. Dimensional reduction of topological or cohomological field theories is equivalent to choosing a Euclidean metric, reducing by translations along the coordinate to be ignored, and bringing the action to a covariant form (restricting to the allowed metrics if necessary) [10]. Such reductions to two and three dimensions of the Donaldson–Witten theory have been mentioned already. However, a further reduction of the Donaldson–Witten theory to dimension 1 could be attempted by electing a Euclidean metric or any metric which is invariant under a set of three orthogonal translations with suitable boundary conditions. With a specific choice of gauge (for instance axial) and for Euclidean space, the SDYM equations probed will be reduced to the well known Nahm equations [39], with a corresponding topological quantum mechanical system to be shown later in this paper.

More generally, given a topological or cohomological theory and a choice of metric, reductions by certain subgroups of the invariance group of the system, for example isometries, can also be explored. As for reductions by translational symmetries, a covariant description can be sought, up to allowed metrics for restricted cohomological cases. It is hoped to find cohomological field theories which would probe the moduli space of reduced systems of equations of interest.

In what follows, section 2 provides a short reminder of cohomological quantum field theories, and aspects of the symmetry reduction of topological/cohomological field theories are discussed. Section 3 presents examples of reduction with respect to subgroups including rotational symmetries of the Donaldson–Witten theory and a cohomological field theory in higher dimensions. Then, the reduction of the Donaldson–Witten theory to a topological quantum mechanical theory, which should probe the moduli space of the Nahm equations, and a set of deformed versions are shown in section 4. Section 5 includes a summary and possible developments.

2. Reduction

Let us recall some general aspects of topological quantum field theories (TQFTs) and simultaneously set our notation. A TQFT of the Witten type (or cohomological type) [20] is described by a partition function (Z):

$$Z = \int \mathrm{d}\Phi\, \mathrm{e}^{-S_q} \tag{2.1}$$

where Φ denotes the even and odd dynamical fields, ghosts, anti-ghosts and multiplier fields. All fields are defined on a Riemannian manifold M of dimension n with metric g. The quantum action S_q is determined by a BRST operator Q (assumed to be metric independent) and a functional $V(\Phi, g)$ according to

$$S_q = \{Q, V(\Phi, g)\}. \tag{2.2}$$

Physical states correspond to Q-cohomology classes.

Under metric variations δg, the quantum action obeys

$$\delta_g S_q = \{Q, \delta_g V\} = \frac{1}{2}\int_M \mathrm{d}^n x\, \sqrt{g} T_{\alpha\beta}\delta g^{\alpha\beta} \tag{2.3}$$

where

$$T_{\alpha\beta} = \{Q, V_{\alpha\beta}(\Phi, g)\} := \{Q, \delta V/\delta g^{\alpha\beta}\}. \tag{2.4}$$

Assuming no metric anomalies, the variation of the partition function with respect to the metric vanishes because $\delta_g S$ is Q-exact:

$$\delta_g Z = \langle \delta_g S_q \rangle = \langle \{Q, \delta_g V\} \rangle = 0. \tag{2.5}$$

Moreover, for a general operator $\mathcal{O}$,

$$\delta_g \langle \mathcal{O} \rangle = \langle \delta_g \mathcal{O} - \mathcal{O}\delta_g S_q \rangle. \tag{2.6}$$

Hence, the variation vanishes non-trivially for Q-cohomology classes of operators $\mathcal{O}$ satisfying $\delta_g \mathcal{O} = \{Q, \tilde{\mathcal{O}}\}$ for some operator $\tilde{\mathcal{O}}$. This metric independence is the hallmark of TQFTs.

Closely related to TQFTs are what will be called below cohomological quantum field theories (CQFTs). As with TQFTs, CQFTs are defined by the cohomology of a BRST operator: however, they are not necessarily independent of the metric on the target manifold. Consequently, a CQFT is best characterized by its Q-cohomology.

If a set of operators $\mathcal{O}^{(0)}, \mathcal{O}^{(1)}, \ldots, \mathcal{O}^{(n)}$, satisfy a set of topological descent equations

$$0 = \{Q, \mathcal{O}^{(0)}\}; -d\mathcal{O}^{(0)} = \{Q, \mathcal{O}^{(1)}\}; \ldots; -d\mathcal{O}^{(n-1)} = \{Q, \mathcal{O}^{(n)}\}; -d\mathcal{O}^{(n)} = 0 \tag{2.7}$$

then it is well known that BRST invariant observables can be obtained by arbitrary products of the following BRST invariant operator functionals:

$$W^{(i)}(\gamma) = \int_{\gamma_i} \mathcal{O}^{(i)} \tag{2.8}$$

where γ_i is any i-cycle in homology, and $i \leqslant n$.

Now, within the context of a TQFT/CQFT, let us suppose that there exists an invariance symmetry group T acting on M through the map $\sigma : T \times M \to M$. T-invariant gauge fields A_μ, or connections ($\omega = A_\mu \theta^\mu$, with co-frame basis θ^μ, $\mu = 1, \ldots, n$), are required to respect the following global invariance conditions [41, 42]:

$$\sigma_t^* \omega = Ad\rho^{-1}(t, x)\omega + \rho^{-1}(t, x)d\rho(t, x) \tag{2.9}$$

where $t \in T$, $x \in M$, and $\rho : T \times M \to H$ (gauge group) is a transformation function which characterizes the lift of the σ-action to the gauge bundle.

If the transformation function $\rho(t, x) = Z(H)$, where $Z(H)$ denotes the centre of H, the invariance condition becomes locally [40–43]:

$$L_{X_t}\omega = 0 \tag{2.10}$$

where L_{X_t} stands for the Lie derivative with respect to the vector field X_t which corresponds to the element of the Lie algebra associated with the group element $t \in T$. Nevertheless, if the curvature F of the connection ω is nonvanishing and span a two-dimensional subspace of $\Lambda^2(T^*M)$ at $x \in M$, then the most general invariant generic gauge field (or connection) can be identified as irreducible at x for transitive actions on M if the image of the homorphism

$\lambda_* : \mathcal{T}_o \to \mathcal{H}$, from the isotropy subalgebra $\mathcal{T}_o$ at x to the gauge algebra $\mathcal{H}$, vanishes. This result follows from the compatibility of the (local) irreducibility equation: $D_\mu \xi = 0$. The strict invariance [41,42] of a gauge field (from a trivial lift of the gauge symmetry group action to the gauge bundle) is in general an indication of the presence of an irreducible gauge field ($h^{-1}A_\mu h + h^{-1}\partial_\mu h = A_\mu$, provided $h \in Z(H)$).

If the infinitesimal invariance of any field ϕ of the set Φ under the action of any element $t \in T$ satisfies the equation

$$\delta_t \phi = 0 \tag{2.11}$$

and if the BRST (Q) transformation can be written in terms of a function f of the set of fields Φ with possibly their first-order derivatives

$$\delta_Q \phi = f(\Phi, \partial_\mu \Phi) \tag{2.12}$$

it then follows that

$$[\delta_t, \delta_Q]\phi = 0 \tag{2.13}$$

for any $\phi \in \Phi$ and $t \in T$ if

$$\delta_t f(\Phi, \partial_\mu \Phi) = 0. \tag{2.14}$$

We note that (2.14) holds for a T-invariant BRST charge, i.e. for $\delta_t Q = 0$.

The presence of linear derivative terms in (2.12) restricts symmetries allowing reduced or residual BRST transformations. For these terms, the vanishing commutator,

$$[X_t, \partial_\mu]\Phi = \sum_\alpha F^\alpha(x) X_{\tilde{t}_\alpha} \Phi = 0 \qquad \forall \mu \tag{2.15}$$

has to be satisfied in order to fulfil the above condition (2.14) on $f(\Phi, \partial_\mu \Phi)$. Here, $\tilde{t}_\alpha$ is any element of T and could vary for different t. $F^\alpha(x)$ are functions of the independent coordinates with $\alpha \leqslant$ (dimension of the symmetry algebra $\mathcal{T}$). Equivalently, it could be stated that the algebra of translations generated by $\{\partial_\mu\}$ has, modulo the symmetry algebra, to commute with the symmetry algebra.

Relation (2.13) implies the existence of a residual BRST transformation on the set of T-invariant Φ. Therefore, the quantum action has to be T-invariant:

$$\delta_t S_q = \{Q, \delta_t V\} = 0. \tag{2.16}$$

Accordingly, assuming no T-symmetry anomaly, the topological invariant $\langle \mathcal{O} \rangle$ is also T-symmetric:

$$\delta_t \langle \mathcal{O} \rangle = \langle \delta_t \mathcal{O} - \mathcal{O} \delta_t S_q \rangle = 0 \tag{2.17}$$

for any $t \in T$.

Using the T-invariant fields Φ^R and BRST charge Q^R, a reduced partition function Z^R can be defined:

$$Z^R = \int \mathrm{d}\Phi^R \, \mathrm{e}^{-S_q^R} \tag{2.18}$$

where S_q^R is the reduced quantum action such that

$$S_q^R = \{Q^R, V^R(\phi^R, g^R)\} \tag{2.19}$$

for the reduced metric g^R and functional V^R, with an integration over the manifold spanned by the T-invariant independent variables. These definitions involve residual fields and also allow us to find reduced operators, or observables, denoted $\mathcal{O}^R$, as well as possibly nontrivial topological invariants $\langle \mathcal{O}^R \rangle$.

A correspondence with the topological invariants $\langle\mathcal{O}\rangle$ can be described via a relation between the reduced fields Φ^R, their complements in the space of fields, denoted by $\Phi^\perp$, and the original set of fields Φ. Let us suppose that there is a well defined transformation $\Phi = \Phi(\tilde{\Phi})$, where $\tilde{\Phi}$ stands for the complete set of fields $(\Phi^R, \Phi^\perp)$. Then, the topological invariants can be written as

$$\langle\mathcal{O}\rangle = \int \mathrm{d}\tilde{\Phi} \left|\mathrm{sdet}\left(\frac{\delta\Phi}{\delta\tilde{\Phi}}\right)\right| \mathrm{e}^{-S_q(\Phi(\tilde{\Phi}))}. \tag{2.20}$$

When 'sdet' is independent of the reduced fields Φ^R, one can write

$$\langle\mathcal{O}\rangle = \int \left|\mathrm{sdet}\left(\frac{\delta\Phi}{\delta\tilde{\Phi}}\right)\right| \mathrm{d}\Phi^\perp \langle\mathcal{O}^R\rangle. \tag{2.21}$$

To conclude this section, the reduction of a TQFT/CQFT, with respect to a suitably restricted invariance symmetry group, leads to a new TQFT/CQFT with a T-invariant BRST charge and T-invariant fields defined on a reduced manifold. The allowed reductions include dimensional reduction, but more general reductions are possible.

3. Examples

In this section, for simplicity, we consider the Donaldson–Witten theory on a manifold with the Euclidean group as isometry group. An example of reduction involving a subgroup of the Euclidean group is presented.

It is known that the Donaldson–Witten theory can be derived using the Langevin approach starting with the classical action (S_c) [18]

$$S_c = \tfrac{1}{2}\int_{M^4} \mathrm{d}^4x\,\sqrt{g}\,\mathrm{tr}(G_{\alpha\beta} - F^+_{\alpha\beta})^2 \tag{3.1}$$

where $F^+_{\alpha\beta} = \frac{1}{2}(F_{\alpha\beta} + \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}F^{\mu\nu})$. Subsequent application of the Batalin–Vilkovisky method leads to the following complete quantum action (S_q):

$$S_q = -\tfrac{1}{4}\int_{M^4} \mathrm{d}^4x\,\sqrt{g}F_{\alpha\beta}\tilde{F}^{\alpha\beta} + \{Q, V\} \tag{3.2}$$

where

$$V = \int_{M^4} \mathrm{d}^4x\,\sqrt{g}\chi^{\alpha\beta}\left(F^+_{\alpha\beta} - \frac{\alpha}{2}B_{\alpha\beta}\right) + \bar{\phi}(D_\alpha\psi^\alpha) + \bar{c}(\partial_\alpha A^\alpha). \tag{3.3}$$

The gauge choices elected in the above quantization are

$$\partial_\alpha A_\alpha = 0 \qquad D_\alpha\psi^\alpha = 0 \qquad \text{and} \qquad G_{\alpha\beta} = 0. \tag{3.4}$$

The complete set of off-shell nilpotent BRST transformations is given by [18]

$$\begin{aligned} &\{Q, A_\alpha\} = D_\alpha c + \psi_\alpha \qquad \{Q, \psi_\alpha\} = -[c, \psi_\alpha] - D_\alpha\phi \qquad \{Q, \phi\} = -[c, \phi] \\ &\{Q, \chi_{\alpha\beta}\} = B_{\alpha\beta} \qquad \{Q, B_{\alpha\beta}\} = 0 \qquad \{Q, \bar{\phi}\} = \eta \\ &\{Q, c\} = -\tfrac{1}{2}[c, c] + \phi \qquad \{Q, \bar{c}\} = b \qquad \{Q, \eta\} = 0 \qquad \{Q, b\} = 0. \end{aligned} \tag{3.5}$$

Upon reduction with respect to a subalgebra of the isometry algebra spanned by the basis $\{P_3 = \partial_3, P_4 = \partial_4, M_{34} = (x^3\partial_4 - x^4\partial_3)\}$, a residual action corresponding to a two-dimensional topological Yang–Mills theory [44] is obtained by substitution of the invariant set of fields:

$$\begin{array}{ll}
A_\alpha = (A_1(x^1, x^2), A_2(x^1, x^2), 0, 0) & \psi_\alpha = (\psi_1(x^1, x^2), \psi_2(x^1, x^2), 0, 0) \\
c = c(x^1, x^2) \qquad \phi = \phi(x^1, x^2) & \\
\bar{c} = \bar{c}(x^1, x^2) \qquad b = b(x^1, x^2) & \\
\eta = \eta(x^1, x^2) \qquad \bar{\phi} = \bar{\phi}(x^1, x^2) & \\
\chi_{12}(x^1, x^2) = -\chi_{21}(x^1, x^2) = \chi_{34}(x^1, x^2) = -\chi_{43}(x^1, x^2) & \text{otherwise} \quad 0 \\
B_{12}(x^1, x^2) = -B_{21}(x^1, x^2) = B_{34}(x^1, x^2) = -B_{43}(x^1, x^2) & \text{otherwise} \quad 0.
\end{array} \tag{3.6}$$

They provide the reduced off-shell BRST transformations:

$$\begin{array}{llll}
\{Q, A_m\} = D_m c + \psi_m & \{Q, \psi_m\} = -[c, \psi_m] - D_m\phi & \{Q, \phi\} = -[c, \phi] & \\
\{Q, \chi_{mn}\} = B_{mn} & \{Q, B_{mn}\} = 0 & \{Q, \bar{\phi}\} = \eta & \\
\{Q, c\} = -\frac{1}{2}[c, c] + \phi & \{Q, \bar{c}\} = b & \{Q, \eta\} = 0 & \{Q, b\} = 0
\end{array}$$

where the indices $m, n = 1, 2$.

As further illustration, the $D = 8$ CQFT of [8] could be reduced under a subgroup of the Euclidean isometry group associated with the subalgebra $\{P_7 = \partial_7, P_8 = \partial_8, M_{78} = (x^7\partial_8 - x^8\partial_7)\}$. This would lead to a six-dimensional BRST invariant action. In a manner similar to the previous example, the invariant fields would be substituted in the original action and BRST transformations. The invariant self-dual (antisymmetric) field $\chi_{\alpha\beta}$ obeys the octonionic related equations

$$\chi_{8a} = \tfrac{1}{2} c_{abc} \chi_{bc} \tag{3.7}$$

where the c_{abc}, $a, b, c = 1, \ldots 7$, correspond to the octonionic structure constants [8, 27]. A further reduction to four dimensions could also be carried out by imposing the invariance of the six-dimensional fields with respect to the subgroup with algebra basis $\{P_5 = \partial_5, P_6 = \partial_6, M_{56} = (x^5\partial_6 - x^6\partial_5)\}$.

In the above two reductions from eight dimensions, the residual covariance is $SO(6)$ and $SO(4)$ respectively. The residual holonomy is determined by the subgroup of $SO(6)$ (resp. $SO(4)$) preserving the Spin(7) invariant antisymmetric tensor, $T_{\mu\nu\rho\sigma} = \eta^T \gamma^{\mu\nu\rho\sigma} \eta$, where $\gamma^{\mu\nu\rho\sigma}$ is the completely antisymmetric product of the γ matrices associated to $SO(8)$, and η is the covariantly constant spinor field normalized to unity leading to a division of the space of chiral real majorana spinors on M^8. For the four-dimensional reduction, the $T_{\mu\nu\rho\sigma}$ tensor is easily found to be reduced to the usual $SO(4)$ invariant completely antisymmetric tensor.

The residual equations probed are obtained from the octonionic equations,

$$F_{8a} = \tfrac{1}{2} c_{abc} F_{bc} \tag{3.8}$$

by inserting the vanishing components of the invariant $F_{\mu\nu}$ in equation (3.8). It can be verified that the $D = 4$ SDYM equations are left after reduction with respect to the subgroup with algebra spanned by $\{P_5, P_6, P_7, P_8, M_{56}, M_{78}\}$.

In the following section, a more detailed example of reduction related to integrable systems is presented.

4. Nahm equations

The Nahm equations are known to arise as a set of equations whose solutions provide the monopole solutions, when endowed with suitable boundary conditions [45–47]. Explicit solutions have been reported in a number of papers (see [47–49] and [50], as well as references therein for examples). They involve, in general, theta functions of curves of high genus. The Nahm equations have been encountered in the study of D-branes in $D = 4$, $N = 4$ SYM theory [51], and higher-dimensional generalizations have also been found in relation with higher-dimensional SDYM equations (see [8, 52] for examples).

As mentioned previously, and similarly to a dimensional reduction, the Donaldson–Witten theory has been reduced by one-and two-dimensional translational symmetry groups to derive three- and two-dimensional TQFTs respectively, which correspondingly would describe reduced SDYM equations solution sets. A natural extension is to effect a reduction of the Donaldson–Witten theory under a three-dimensional group of translations.

However, let us recall that the gauge of the fields A_μ and ψ_μ for the previous reductions to two and three dimensions were chosen to satisfy

$$\partial_\mu A^\mu = 0 \qquad \text{and} \qquad D_\mu \psi^\mu = 0. \tag{4.1}$$

For the three-dimensional reduction, the (gauge) condition for the field ψ_μ is kept, but instead of the Lorentz gauge, an axial gauge, $A_4 = 0$, will be selected for the gauge field A_μ (see also [53]). Such a gauge has already been used in the reduction of the Donaldson–Witten theory to Floer theory [15, 18], as well as in its Nicolai map interpretation [54].

The reduced BRST transformations under the subgroup of the isometry group of Euclidean space in four dimensions generated by $\{P_1, P_2, P_3\}$, where P_i denotes the generator of translations along the coordinate x_i, have the form [18]:

$$\begin{aligned}
&\delta A_i = -\epsilon([A_i, c] + \psi_i) \qquad \delta A_4 = -\epsilon(D_4 c + \psi_4)\\
&\delta\psi_i = \epsilon([A_i, \phi] + \{c, \psi_i\}) \qquad \delta\psi_4 = \epsilon(D_4\phi + \{c, \psi_4\})\\
&\delta c = \epsilon(-\phi + \tfrac{1}{2}\{c, c\}) \qquad \delta\phi = \epsilon[c, \phi]\\
&\delta\chi_{\alpha\beta} = \epsilon B'_{\alpha\beta} \qquad \delta B'_{\alpha\beta} = 0\\
&\delta\bar{c} = -\epsilon b \qquad \delta b = 0\\
&\delta\bar{\phi} = -\epsilon\eta, \qquad \delta\eta = 0\\
&\delta G_{ij} = \epsilon([c, G_{ij}] - \tfrac{1}{2}[A_i, \psi_j] + \tfrac{1}{2}[A_j, \psi_i] - \tfrac{1}{4}\epsilon_{ijk}([A_k, \psi_4] - D_4\psi_k) + [\phi, \chi_{ij}])\\
&\delta G_{i4} = \epsilon([c, G_{i4}] - \tfrac{1}{2}[A_i, \psi_4] + \tfrac{1}{2}D_4\psi_i - \tfrac{1}{2}\epsilon_{ijk}[A_j, \psi_k] + [\phi, \chi_{i4}])
\end{aligned} \tag{4.2}$$

which share nilpotency off-shell, where $\delta = -\epsilon\{Q,\ \}$.

Using symmetry reduction on the action (cf [18]), the $\{P_1, P_2, P_3\}$ subgroup reduction leads to the residual quantum action

$$S_q = \int \mathrm{d}x_4\, \mathrm{tr}\{Q, \chi^{\alpha\beta}\left(F^+_{\alpha\beta} - \frac{\alpha}{2}B_{\alpha\beta}\right) + \bar{c}\eta_\alpha A^\alpha + \bar{\phi}D_\alpha\psi^\alpha\}. \tag{4.3}$$

Observables W_l [15, 18] are l-forms on the four-dimensional manifold, which are reduced to 0- and 1-forms on R or S^1:

$$\begin{array}{ll}
\text{0-forms} & \text{1-forms}\\
W_0 = \frac{1}{2}\,\mathrm{tr}(\phi^2) & 0\\
W_{1_i} = \mathrm{tr}(\phi\psi_i) & W_{1_4} = \mathrm{tr}(\phi\psi_4)\\
W_{2_{ij}} = -\,\mathrm{tr}(\psi_i\psi_j - \phi F_{ij}) & W_{2_{i4}} = -\,\mathrm{tr}(\psi_i\psi_4 - \phi F_{i4})\\
W_{3_{ijk}} = \frac{1}{9}\,\mathrm{tr}(\psi_k F_{ij} + \text{cyclic}) & W_{3_{ij4}} = \frac{1}{9}\,\mathrm{tr}(\psi_4 F_{ij} + \text{cyclic})\\
0 & W_{4_4} = \frac{4}{3}\,\mathrm{tr}(\epsilon_{ijk}F_{ij}F_{k4})
\end{array} \tag{4.4}$$

where the indices $i, j, k = 1, 2, 3$. They satisfy the following BRST transformation relations (or topological descent equations):

$$\begin{array}{ll} \text{0-forms} & \text{1-forms} \\ \{Q, W_0\} = 0 & \\ \{Q, W_{1_i}\} = 0 & \{Q, W_{1_4}\} = -dW_0 \\ \{Q, W_{2_{ij}}\} = 0 & \{Q, W_{2_{i4}}\} = -dW_{1_i} \\ \{Q, W_{3_{ijk}}\} = 0 & \{Q, W_{3_{ij4}}\} = -dW_{2_{ij}} \\ & \{Q, W_{4_4}\} = -d(\epsilon_{ijk} W_{3_{ijk}}) \\ & 0 = dW_{4_4}. \end{array} \tag{4.5}$$

Their integration over corresponding 0- and 1 homology cycles on R or S^1 allow to find functionals which constitute topological invariants:

$$W_\alpha^{(0)}(\gamma) = \int_\gamma W_\alpha^{(0)} \qquad \text{and} \qquad W_\alpha^{(1)}(\gamma) = \int_\gamma W_\alpha^{(1)} \tag{4.6}$$

where $W_\alpha^{(0)}$, and $W_\alpha^{(1)}$ stand respectively for 0- and 1-forms on R or S^1 defined by equations (4.4).

Similarly to the four-dimensional case, nontrivial residual observables could lead to topological invariants [15, 18].

Recall that the gauge groups of rank $\geqslant 2$ can also be considered, but the presence of reducible gauge connections poses difficulties [15, 16, 23, 56].

These observables and topological invariants could also be well defined and nontrivial for 'deformed' forms of the Nahm equations, which are generated through a metric background depending only on the invariant variable (x_4) with respect to the action of the subgroup associated with the basis algebra $\{P_1, P_2, P_3\}$. For a given background metric g with $e = \sqrt{|g|}$, the (anti-)SDYM equations have the form

$$F_{\mu\nu} + \frac{e}{2}\epsilon_{\mu\nu\lambda\sigma} g^{\lambda\lambda'} g^{\sigma\sigma'} F_{\lambda'\sigma'} = 0. \tag{4.7}$$

If $g = g(x^4)$, their reduction under the symmetry group of translations spanned by $\{P_1, P_2, P_3\}$ will produce a set of coupled ODEs which can be called 'deformed' Nahm equations.

The moduli space of these new equations would be related to the correspondingly reduced Donaldson–Witten theory under the same translational symmetries. For example, if the metric chosen is diagonal:

$$g = \sum_{i=1}^{4} g_{ii}(x^4)(dx^i)^2 \tag{4.8}$$

the reduction of the 'curved' (anti-)SDYM equations (4.7) can be written as

$$\begin{aligned} (eg^{11}g^{44})\partial_4 A_1 &= [A_2, A_3] \\ (eg^{22}g^{44})\partial_4 A_2 &= [A_3, A_1] \\ (eg^{33}g^{44})\partial_4 A_3 &= [A_1, A_2]. \end{aligned} \tag{4.9}$$

One might then wonder if a similar reduction via translations of the Abelian monopole equations and their quantum field theoretical equivalent would provide a dual description of the above systems.

5. Conclusion/summary

In the above, the reduction by symmetry of topological/cohomological Yang–Mills theories in different dimensions was considered. It has been shown that reduction under certain symmetries, which can be more general than translations, allow the preservation of cohomological characteristics of the original system. A few examples of reduction from known cohomological models were considered, as well as a particular reduction under three-dimensional translational symmetries of the Donaldson–Witten theory related to an integrable system: the Nahm equations. A quantum system with residual BRST transformations, which could explore the moduli space of the Nahm equations, and some deformed cases have been given.

Future research could explore the reduction of the (non-)Abelian monopole equations and the determination of dual descriptions to the residual systems. Different TQFTs associated to other self-duality equations could also be investigated, as well as their moduli spaces. The reduction of the Donaldson–Witten theory to TQFTs linked to other integrable systems of dimension 1, 2, or 3 derivable from the reduction of SDYM equations could be sought. Finally, one could search for an interpretation of these reduced theories within the Mathai–Quillen formalism; and other TQFTs, such as two-dimensional gravity, could be explored.

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